Finite Math A

Chapter 5: Euler Paths and Circuits
The Mathematics of Getting Around

Academic Standards Covered in this Chapter:

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FM.N.1: Use networks, traceable paths, tree diagrams, Venn diagrams, and other pictorial representations to find the number of outcomes in a problem situation

FM.N.2: Optimize networks in different ways and in different contexts by finding minimal spanning trees, shortest paths, and Hamiltonian paths including real-world problems.

FM.N.4: Construct and interpret directed and undirected graphs, decision trees, networks, and flow charts that model real-world contexts and problems.

FM.N.6: Construct vertex-edge graph models involving relationships among a finite number of elements. Describe a vertex-edge graph using an adjacency matrix. Use vertex-edge graph models to solve problems in a variety of real-world settings.

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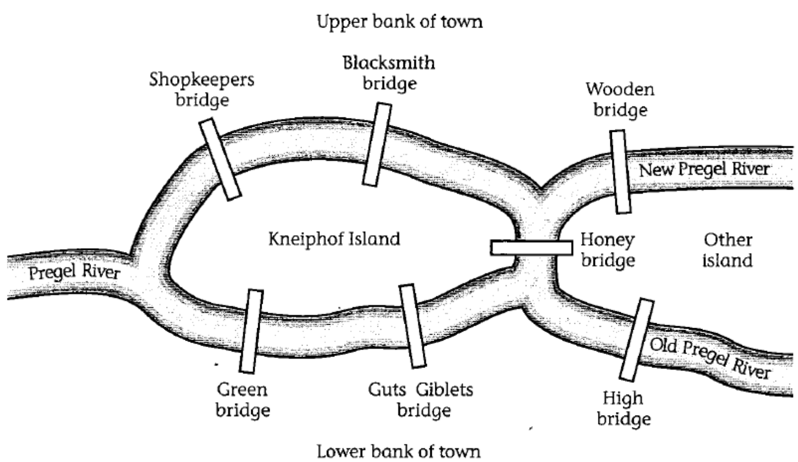
Leonhard Euler
(Pronounced: "Oil-er"
1707-1783
Referred to as the "Shakespeare of Mathematics" and "The Father of Graph Theory" whose collective works and creative output outnumber any mathematician to this day.

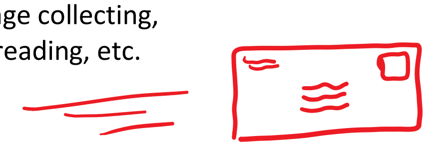


The Beginning of Graph Theory
The Scene: Medieval Town of Köningsberg, Eastern Europe, 1700's
The Puzzle: Can you walk around town in such a way that you cross each bridge once and only once?
The Result: An intrigued Euler's solution lays the foundation for a new branch of mathematics: Graph Theory

5.1 Euler Circuit Problems

Routing Problems: finding ways to route the delivery of goods or services to an assortment of destinations. An Euler circuit problem is a specific type of routing problem where every single street (or bridges, highways, etc) MUST BE COVERED by the route. This is called the exhaustion requirement.

Common Exhaustive Routing Problems: mail delivery, police patrols, garbage collecting, Street sweeping, snow removal, parade routes, tour buses, electric meter reading, etc.



Examples:
The Sunnyside Neighborhood (pg 169)

The Security Guard: A private security guard is hired to make an exhaustive patrol on foot. He parks his car at the corner across from the school (S). Can he start at S and walk every block of the neighborhood JUST ONCE? If not, what is the OPTIMAL trip? How many streets will he have to walk twice?

The Mail Carrier: The mail carrier starts and ends at the post office (P). She must deliver mail on every street there are houses. If there are houses on both sides of the street, she must walk that block twice. She would like to cover the neighborhood with the least amount of walking.

Try and trace this with your pencil. What do you think? Very quickly you run into "overlap" and deadhead travel.

5.2 What is a Graph?

> GRAPH
>
> A **graph** is a structure consisting of a set of objects (the vertex set) and a list describing how pairs of objects are related (the edge set). Relations among objects include the possibility of an object being related to itself (a loop) as well as multiple relations between the same pair of objects (multiple edges).

In plain English: Lines connecting Points

The graphs we discuss in this chapter have no connection to the graphs of functions you've discussed in Algebra.

Ha ha! Math Joke

Vertices: The points on the graph

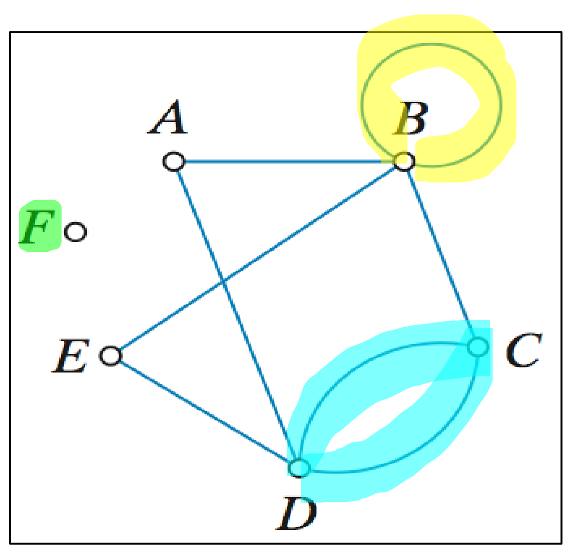
Edges: The lines on the graph.
The edge connecting B and A can be called AB or BA

Loop: An edge connecting a vertex to itself. Example: BB

Multiple Edge: Two edges connecting the same pair of vertices

Example: CD and CD

Isolated Vertices: Vertices not connected by any edges. Example: F

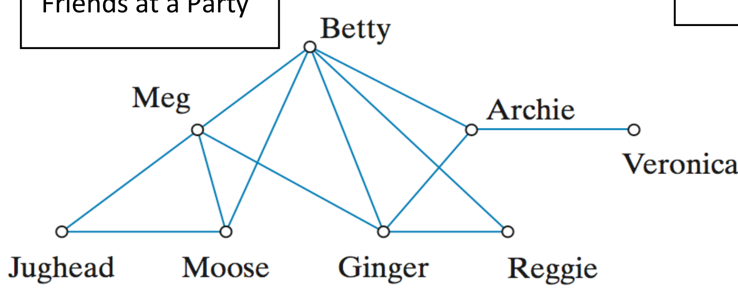


Vertex Set: A, B, C, D, E, F

Edge Set: AB, AD, BB, BC, CD, CD, DE, BE

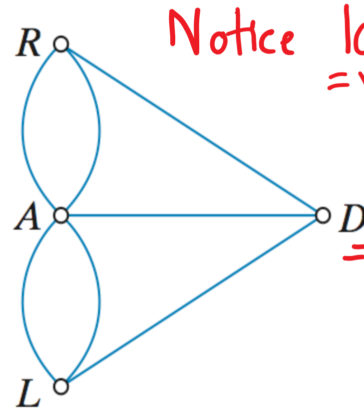
Why are graphs useful?
They help tell a story in a simple picture.

Friends at a Party



who has most friends?
who has least friends?

Euler's Bridge Problem



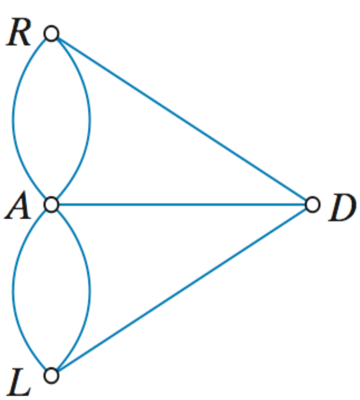
Notice land = vertices
bridges = edges

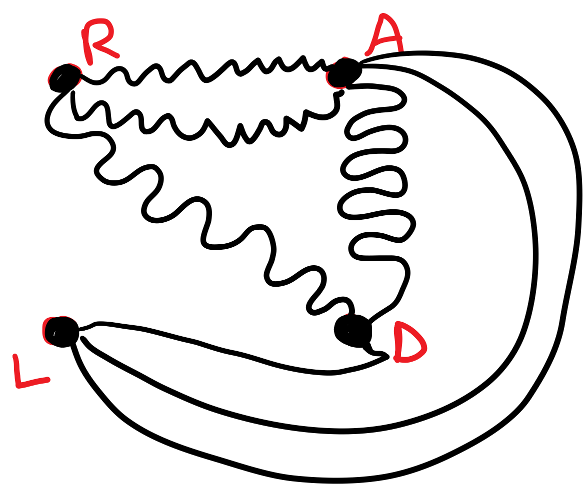
Easier to look at and study

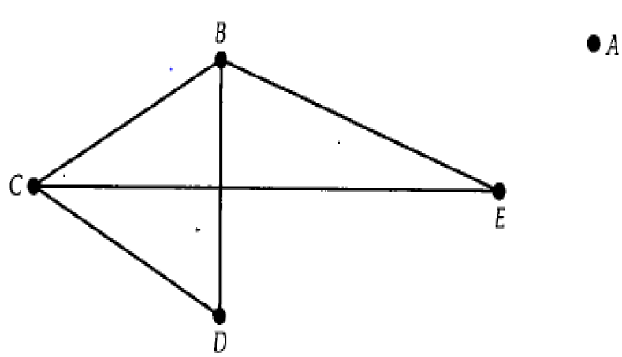
A graph is just a VISUAL REPRESENTATION of a set of objects and the relationships between those objects.

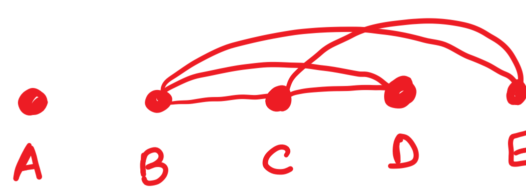
You have a lot of freedom in how you draw a graph model!

Two graphs are ISOMORPHIC if they are exactly the same. That is they have exactly the same vertices and exactly the same edges.

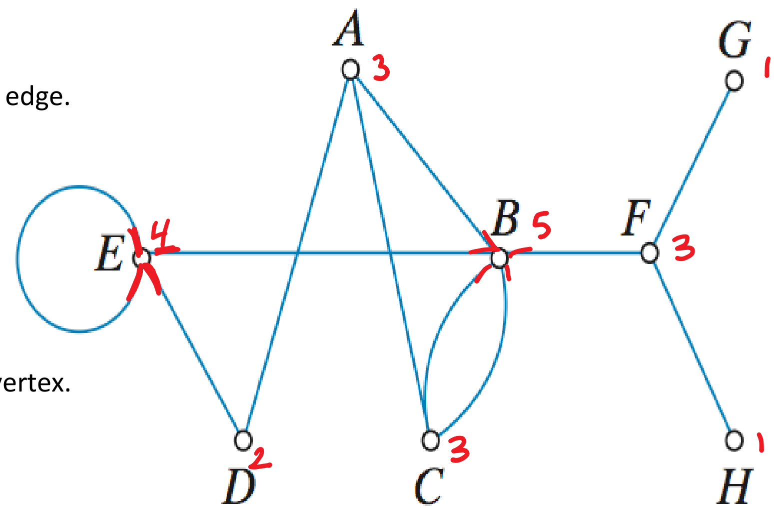








5.3 Graph Concepts and Terminology



Adjacent
Two vertices are adjacent if they are joined by an edge.

Ex: A and B are adjacent.

A and E are not adjacent.

E is adjacent to itself.

Two edges are adjacent if they share a common vertex.

Ex. AB and BF are adjacent.

AB and FG are not adjacent.

Degree deg(V) = #
The number of edges meeting at that vertex.

deg(A) = 3 deg(B) = 5 deg (E) = 4 (loops count as 2 toward the degree)

Vertices whose degree is odd: **"odd vertices"** → A, B, C, F, G, H

Vertices whose degree is even: **"even vertices"** → E, D

Circuit A Closed trip along the edges of the graph.
A circuit starts and ends in the same place.

Path: An Open trip along the edges of the graph.
A path has different starting/ending points.

Important points about circuits and paths:

1. No EDGE can be covered twice.
2. You can use a vertex as many times as you wish.
3. **Length =** the number of edges traveled

Examples of circuits:

Find a circuit through the point B

Yes { BCB, BACB }

~~BFB~~ No!

2

3

Examples of paths:

Find a path from G to E

GFBE

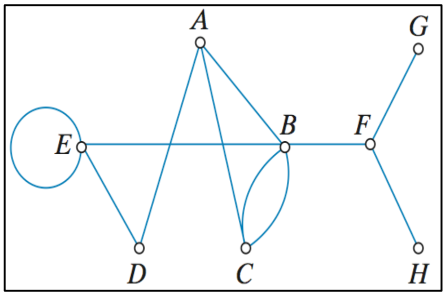
GFBCADE

3

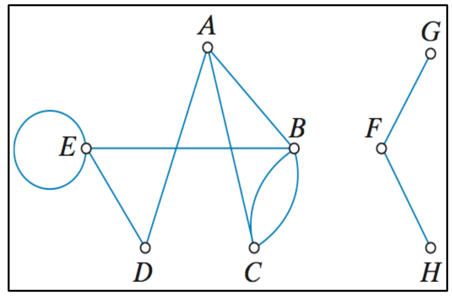
7

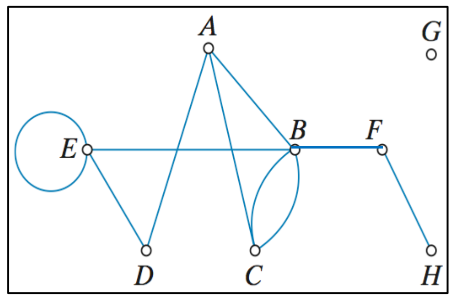
length = # of edges

Connected Think "in ONE piece"...
You can get to any vertex from any vertex along a path.



Disconnected Think "in pieces"....
Graph is made of several components.





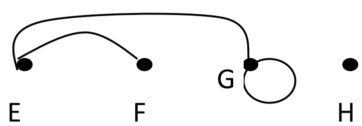
Bridge: An edge that, when removed, causes a connected graph to become disconnected.

In the picture above there are three bridges:

BF, FG, FH

Examples:

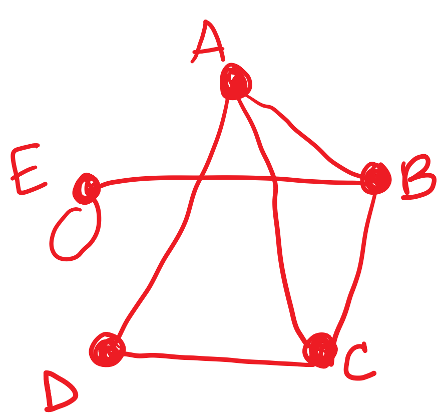
1. For the graph shown, list the degree of each vertex

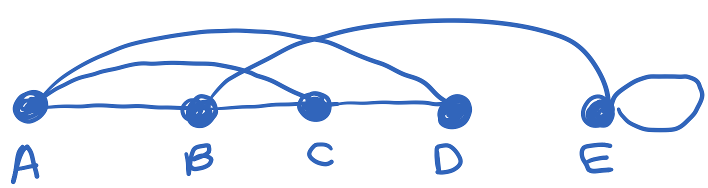


deg(E) = 2 deg(G) = 3
deg(F) = 1 deg(H) = 0

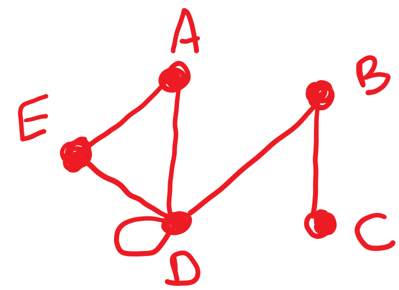
2. Consider the graph with Vertices: A, B, C, D, E,
Edges: AB, AC, AD, BE, BC, CD, EE

Draw two different pictures of the graph. ANSWERS VARY!





3. Consider the graph with Vertices: A,B,C,D,E
Edges: AD, AE, BC, BD, DD, DE



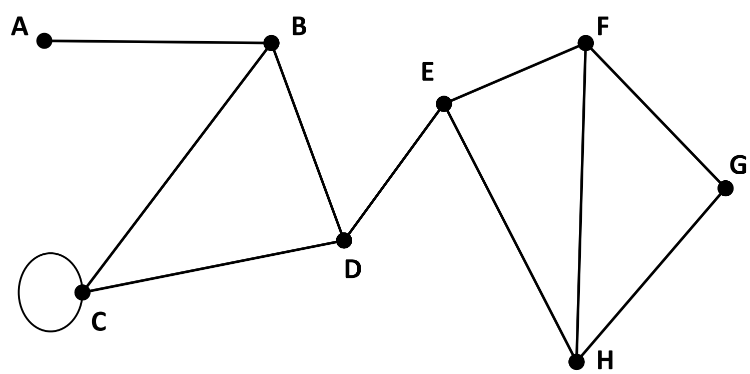
(a) List the vertices adjacent to D.
(b) Find the degree of D
(c) Find the sum of the degrees of the vertices

a) A, B, E, D

b) deg(D) = 5

c) $\underbrace{\deg(A)}_{2} + \underbrace{\deg(B)}_{2} + \underbrace{\deg(C)}_{1} + \underbrace{\deg(D)}_{5} + \underbrace{\deg(E)}_{2} = 12$

4.



a) Find a path of length 4 from A to E

ABCDE

b) Find a circuit through G (answers vary)

GFHG or GHEFG

c) Find a path from A to E that goes through C twice

ABCCDE

d) Find a path of length 7 from F to B

FGHEDCCB

or FHGFEDCB

e) How many paths are there from A to D?

ABD, ABCCD
ABCD

$\boxed{3}$

f) How many paths are there from E to G?

EFG, EFHG
EHG, EHFG

$\boxed{4}$

g) How many paths are there from A to G?

Use multiplication rule

$\boxed{12}$

$\underset{\text{\# ways A to D}}{3} \times \underset{\text{\# ways D to E}}{1} \times \underset{\text{\# ways E to G}}{4}$

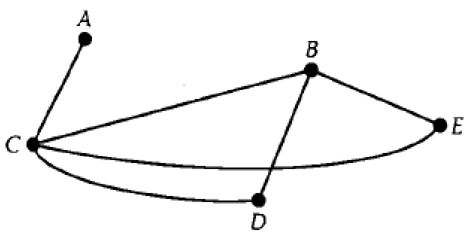
h) Are there any bridges in this graph?

ED and AB

5.Supplement – Adjacency Matrix

There are several ways to describe a graph.

With a diagram:



With a list of vertices/edges:

Vertices = {A, B, C, D, E}

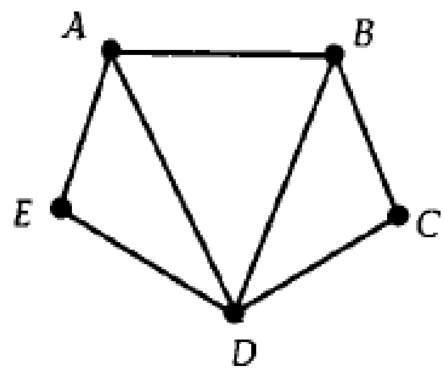
Edges = {AC, CB, CE, CD, BD, BE}.

With an Adjacency Matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

How to create an Adjacency Matrix:

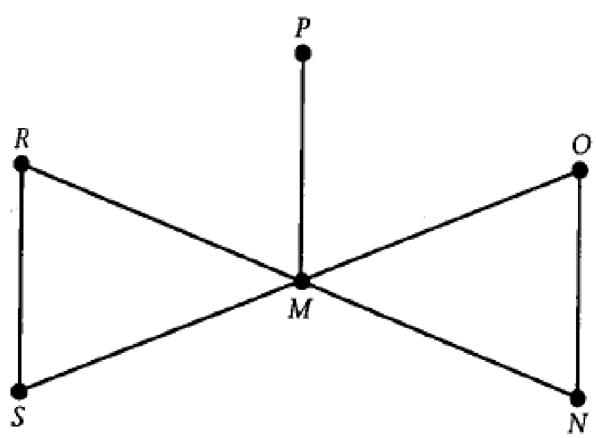
1. Create an n x n square matrix with a row and column for each vertex of your graph.
2. If a vertex is adjacent to another vertex, put a "1" in the corresponding position. If a vertex is *not* adjacent, put a "0" in the corresponding position.



	A	B	C	D	E
A	0	1	0	1	1
B	1	0	1	1	0
C	0	1	0	1	0
D	1	1	1	0	1
E	1	0	0	1	0

Example 1: Make an adjacency matrix for the following graphs

a.

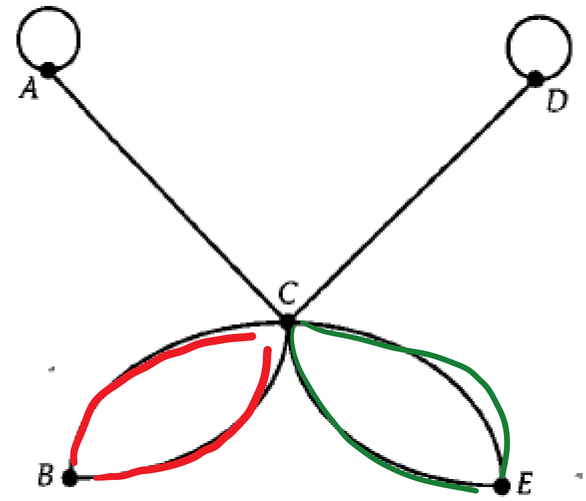


	M	N	O	P	R	S
M	0	1	1	1	1	1
N	1	0	1	0	0	0
O	1	1	0	0	0	0
P	1	0	0	0	0	0
R	1	0	0	0	0	1
S	1	0	0	0	1	0

* Note: It's good form to put your variables in alpha order

* A non-zero # on main diagonal indicates "loop"

b.



	A	B	C	D	E
A	1	0	1	0	0
B	0	0	2	0	0
C	1	2	0	1	2
D	0	0	1	1	0
E	0	0	2	0	0

c. $V = \{E, F, G, J, K, M\}$

$E = \{EF, KM, FG, JM, EG, KJ\}$.

	E	F	G	J	K	M
E	0	1	1	0	0	0
F	1	0	1	0	0	0
G	1	1	0	0	0	0
J	0	0	0	0	1	1
K	0	0	0	1	0	1
M	0	0	0	1	1	0

d. $V = \{W, X, Y, Z\}$

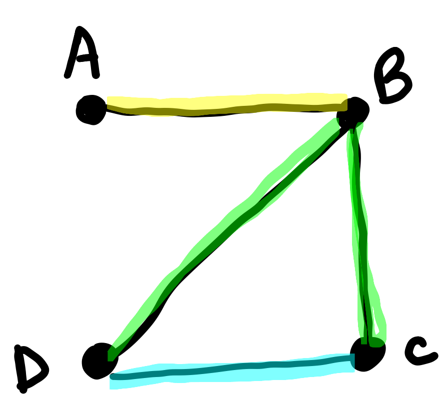
$E = \{WX, XZ, YZ, XY, WZ, WY\}$.

	W	X	Y	Z
W	0	1	1	1
X	1	0	1	1
Y	1	1	0	1
Z	1	1	1	0

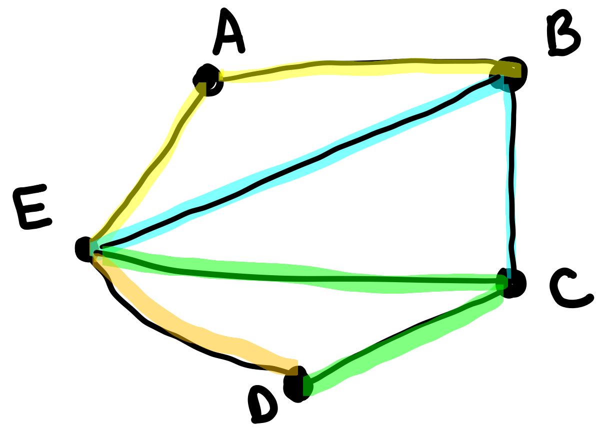
Example 2: Draw the graph given its adjacency matrix

a.

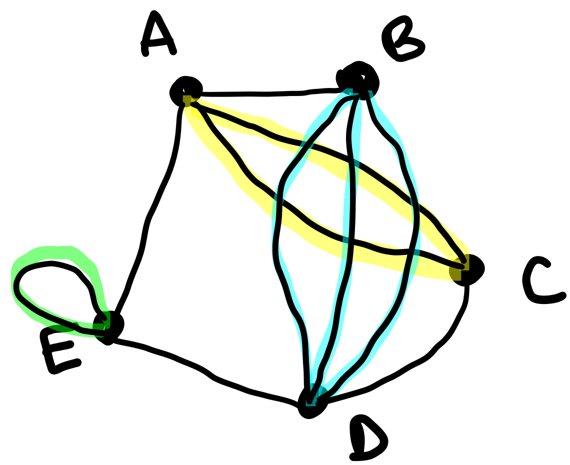
	A	B	C	D
A	0	1	0	0
B	1✔	0	1	1
C	0	1✔	0	1
D	0	1✔	1✔	0



b. $\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$



c. $\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$



<u>**More Graph Terminology**</u>

Simple Graph: No multiple Edges, No loops

Multigraph: Multiple edges, no loops

Pseudograph: Loops, possibly multiple edges also

Loops show up on main diagonal

= Multiedge #>1

Example 3: Which type of graph is represented by the following adjacency matrices?

a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

a) loop, No Multi. edges — **Pseudograph**

b) No loop, No Mult. edge — **Simple Graph**

c) No loop, Multiple edges — **Multigraph**

5.4 Graph Models

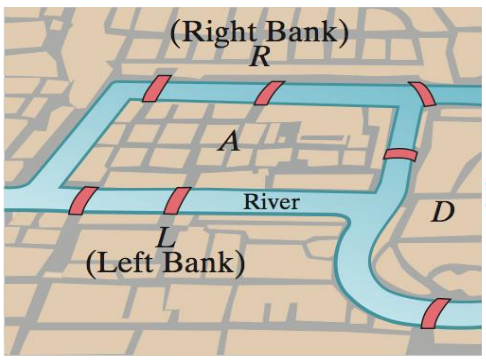
The Goal: Take a real life picture/situation → Create a Graph to Model this

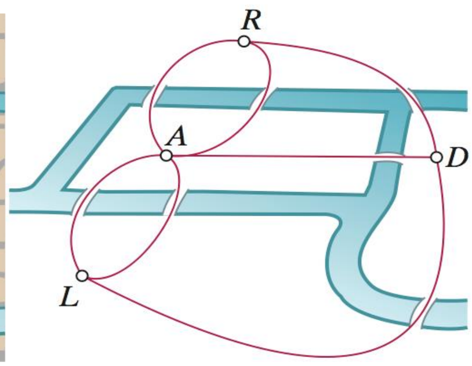
The Rules: Only use vertices (dots) and edges (lines)

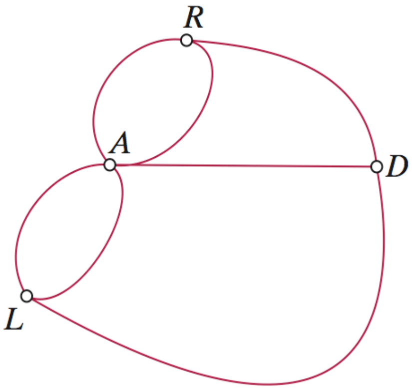
The Result: The only thing that really matters is the relationship between the vertices and the edges.

Examples:

Köningsberg MAP PLAN GRAPH







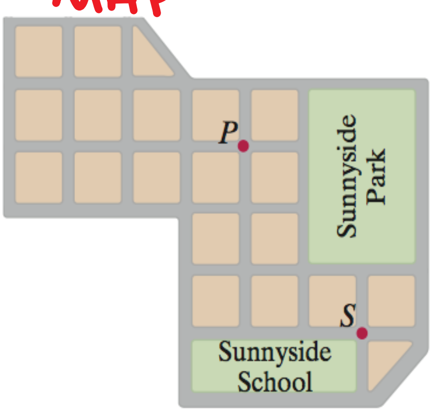
Map

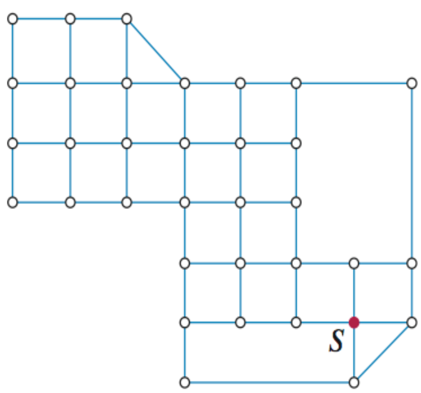
Think: What are my vertices?
What are my edges?

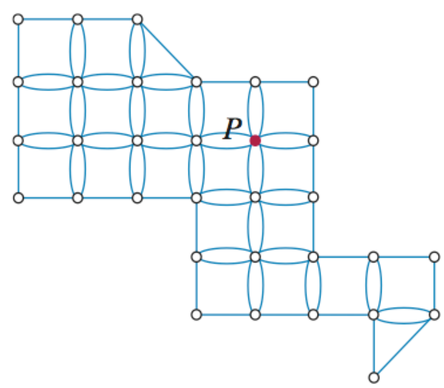
Graph Model

Sunnyside Neighborhood

MAP







Map

Security Guard: Each block once

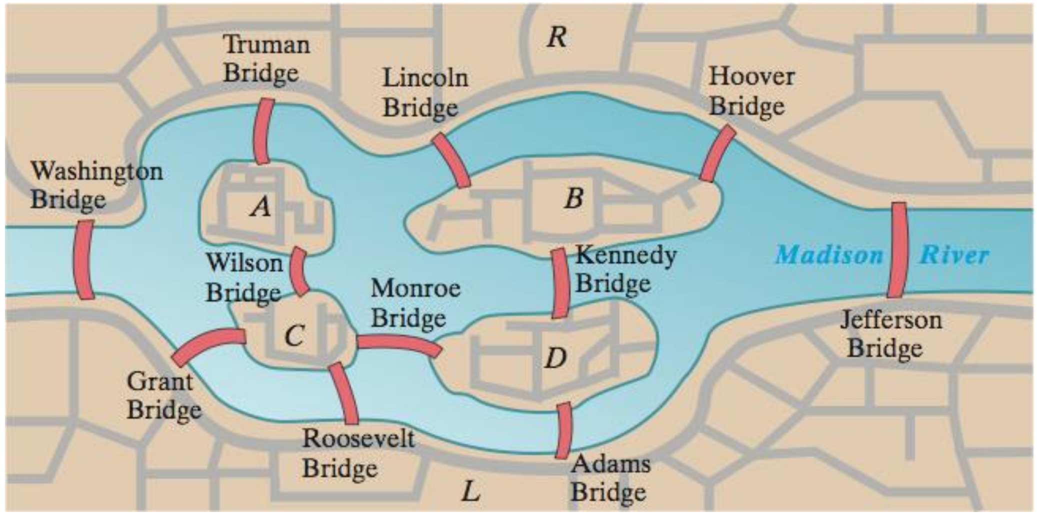
One pass per street

Mail carrier: both sides of any street with houses.

Remember: You are only using vertices and edges!

Vertices may represent: land masses, street intersections, or people
Edges may represent: bridges, streets, or relationships!

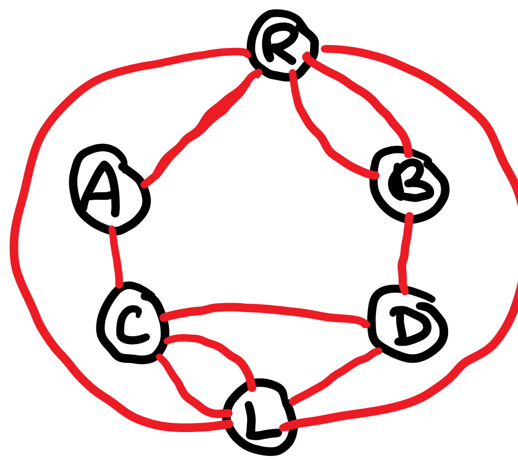
You TRY!
Create a Graph Model



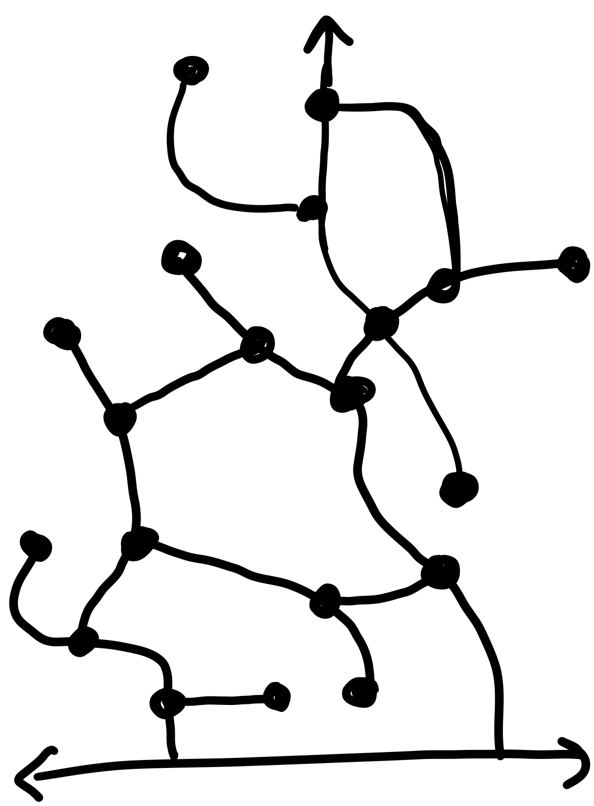
A beautiful river runs through Madison county. There are four islands and 11 bridges joining the islands to both banks of the river (the Right Bank,R, and the Left Bank, L).

Plan: Vertices → land

Edges → bridges



Draw a graph model of this Avon, Indiana neighborhood.



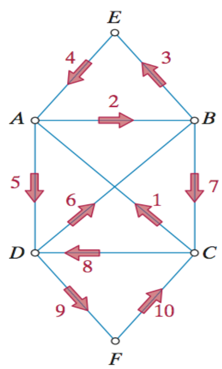
5.5 Euler Theorems

The "Big" Questions

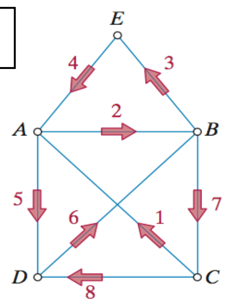
1. When can you draw the figure without retracing any edges and still end up at your starting point?
2. When can you draw the figure without retracing and end up at a point other than the one from which you began?
3. When can you not draw the figure without retracing?

Euler Circuit/Path:
A Circuit/Path that covers EVERY EDGE in the graph once and only once.

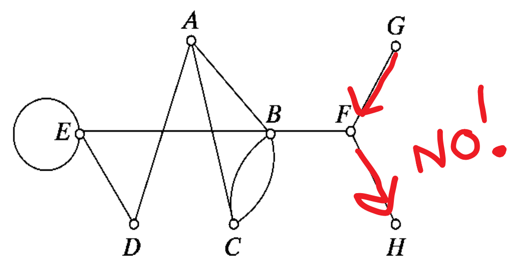
Euler Circuit:

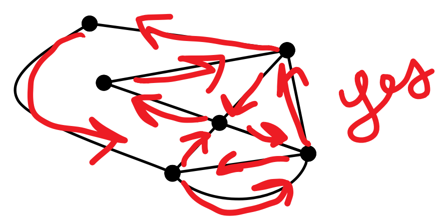


Euler Path:



Do you think that these graphs have an Euler Circuit? Try tracing them with your pencil.





Euler studied a lot of graph models and came up with a simple way of determining if a graph had an Euler Circuit, an Euler Path, or Neither.

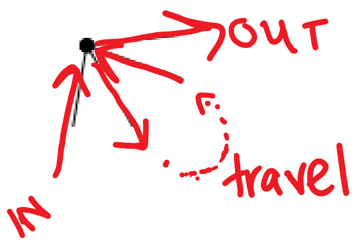
Remember:

Euler Circuit: Travels every edge exactly once, start/end @ same vertex

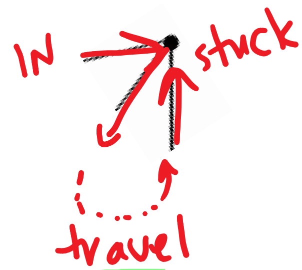
Euler Path: Travels every edge exactly once, start/end @ different vertex

To have an **Euler Circuit**: you must be able to travel IN and OUT of each vertex.

A graph with this vertex would be OK.



A graph with this vertex would NOT!



EULER'S CIRCUIT THEOREM

- If a graph is *connected* and *every vertex is even*, then it has an Euler circuit (at least one, usually more).
- If a graph has *any odd vertices*, then it does not have an Euler circuit.

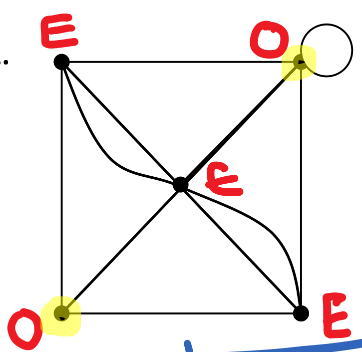
EULER'S PATH THEOREM

- If a graph is *connected* and has *exactly two odd vertices*, then it has an Euler path (at least one, usually more). Any such path must start at one of the odd vertices and end at the other one.
- If a graph has *more than two odd* vertices, then it cannot have an Euler path.

Examples:

Determine if the graph has an Euler Circuit, an Euler Path, or neither and explain why. You do not have to find an actual path or circuit through the graph. COUNT DEGREES!

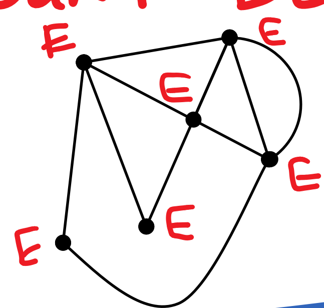
1.



2 odd

Euler Path

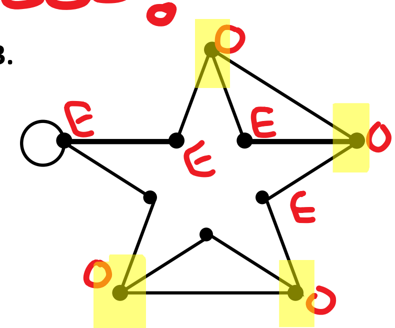
2.



All even

Euler Circuit

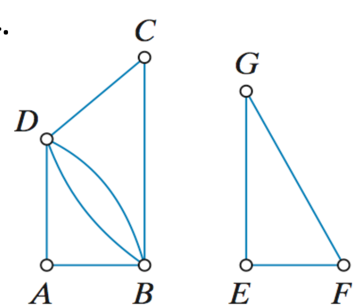
3.



More than 2 odd

Neither

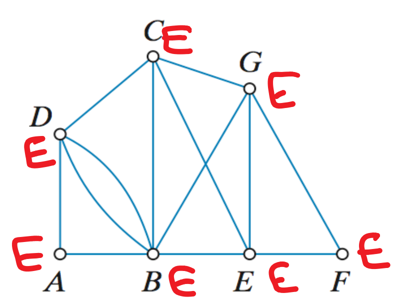
4.



disconnected

Neither

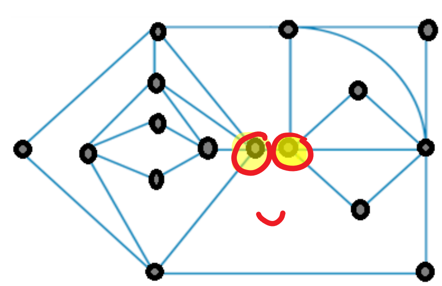
5.



All even

Euler Circuit

6.



Exactly 2 odd

Euler Path

What if there is 1 odd vertex? → Then you counted wrong.

TABLE 5-1 Euler's Theorems (Summary)	
Number of odd vertices	**Conclusion**
0	*G* has Euler circuit
2	*G* has Euler path
4, 6, 8, . . .	*G* has neither
1, 3, 5, . . .	Better go back and double check! This is impossible!

EULER'S SUM OF DEGREES THEOREM

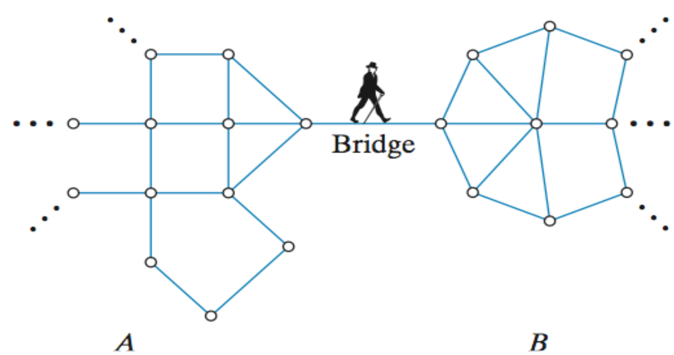
- The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore is an even number).
- A graph always has an even number of *odd* vertices.

5.6 Fleury's Algorithm

Finding an Euler Path or Euler Circuit

Fleury's Algorithm: Don't cross a bridge to an untraveled part of your graph unless you have no other choice.

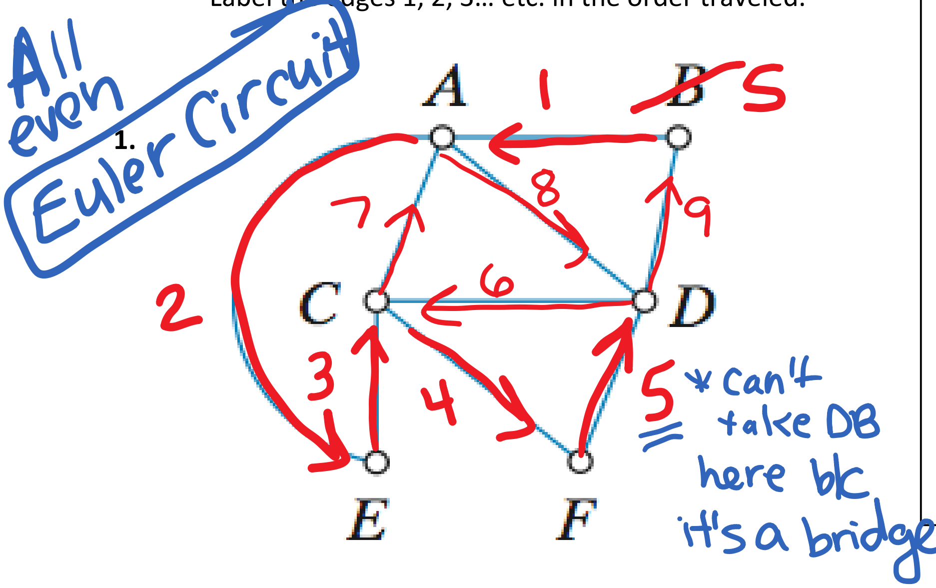
Once you cross the bridge from A to B, the only way to get back to A is by recrossing the bridge which you must not do!!!



Examples:

Does the graph have an Euler Path, Euler Circuit, or Neither? If it has an Euler Path or Euler Circuit, find it! Label the edges 1, 2, 3... etc. in the order traveled.

1.



Steps

1. Make sure the graph is connected
   No odd vertices = Euler circuit
   Two odd vertices = Euler path

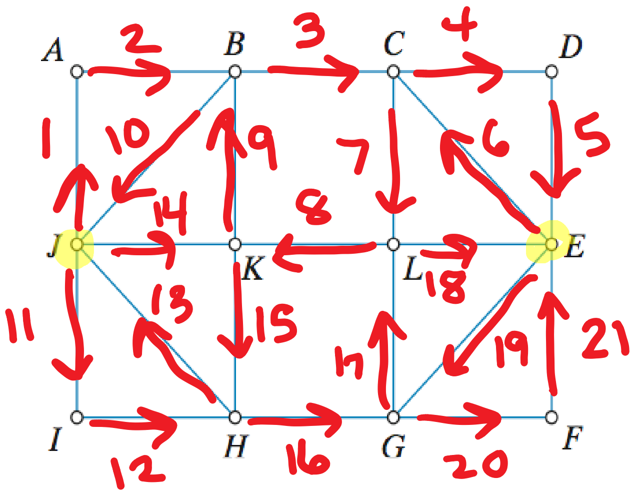
2. Start
   Euler Circuit – start anywhere
   Euler Path – start at an odd vertex

3. If you have a choice, don't choose a bridge. Number your edges as you travel them.

4. When you've traveled every edge exactly once, you're done!
   Euler Circuit – end where you started
   Euler Path – end at the other odd vertex.

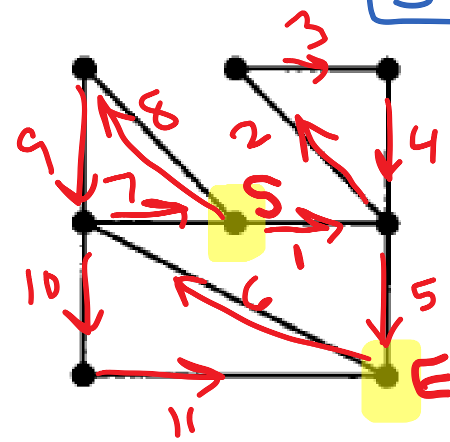
Two odd - $\boxed{\text{Euler Path J to E}}$

2.



Two odd - $\boxed{\text{Euler Path S to E}}$

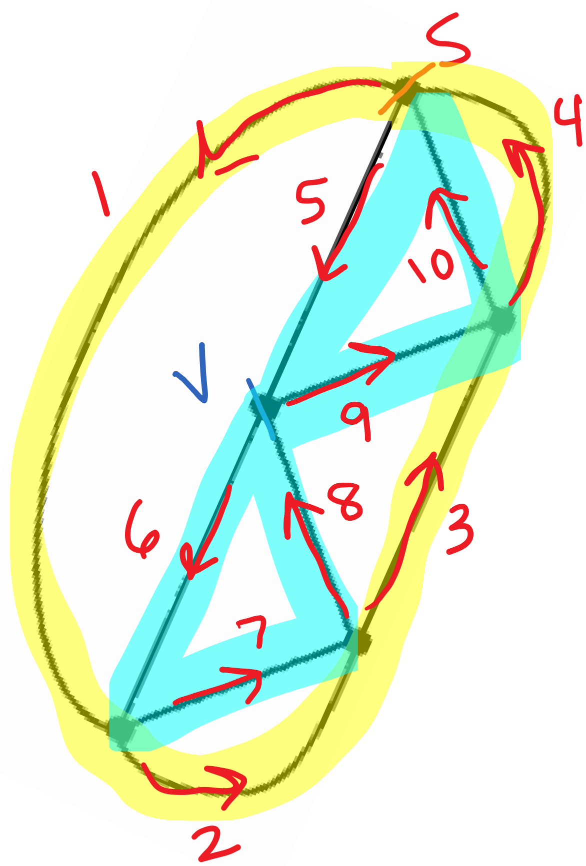
3.



Another Method to try:

You will want multiple colored pens/pencils for this.

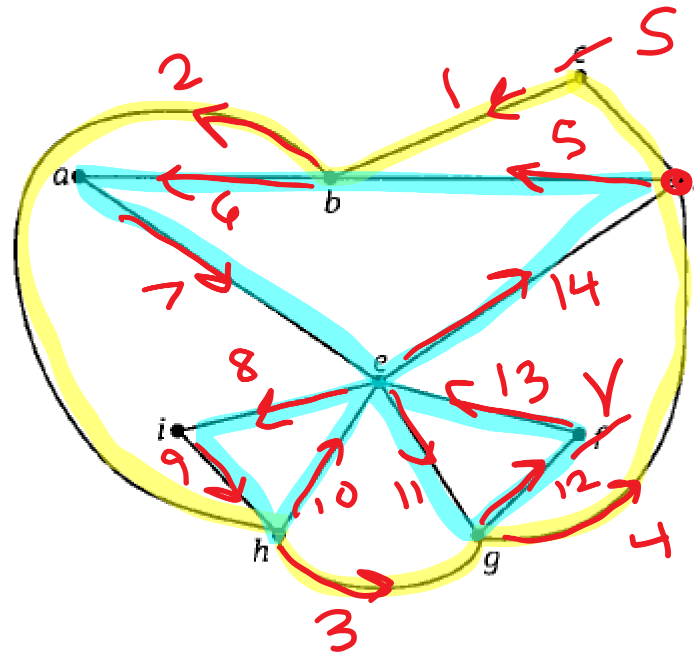
4.



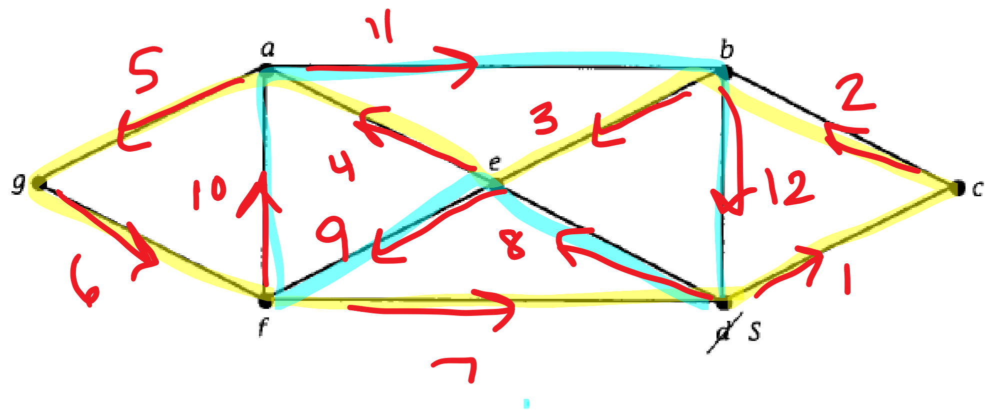
Euler Circuit Algorithm

1. Pick any vertex, and label it S.
2. Construct a circuit, C, that begins and ends at S.
3. If C is a circuit that includes all edges of the graph, go to step 8.
4. Choose a vertex, V, that is in C and has an edge that is not in C.
5. Construct a circuit C' that starts and ends at V using edges not in C.
6. Combine C and C' to form a new circuit. Call this new circuit C.
7. Go to step 3.
8. Stop. C is an Euler circuit for the graph.

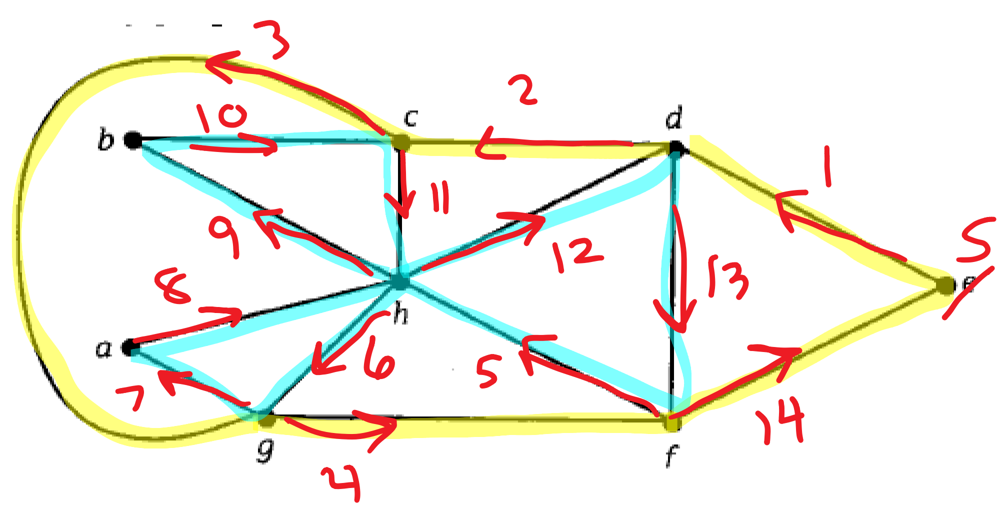
5.


S
c
1
2
5
a
b
d
6
7
14
e
8
13
V
f
i
9
10
11
12
h
g
4
3

6.


11
a
b
5
2
3
g
4
e
c
10
12
9
8
6
1
f
d S
7

7.


3
10
c
2
d
b
1
11
9
12
13
S
8
h
e
a
6
5
7
g
f
14
4

5.7 Eulerizing Graphs

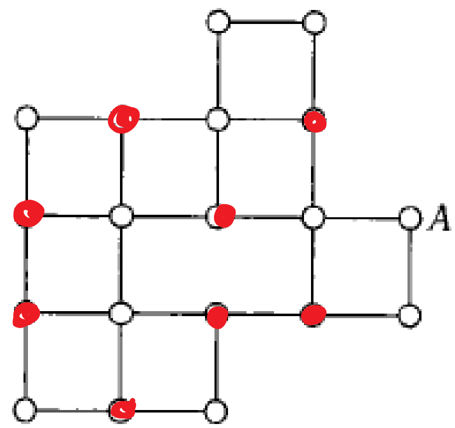
Remember that we are trying to find OPTIMAL SOLUTIONS to routing problems.

We want
1. **an exhaustive route** – covers each and every edge in the graph
2. **to minimize "deadhead" travel** – multiple passes on a single edge is a waste of time/money

OPTIMAL SITUATION:
An Euler Circuit or Path Already Exists

OTHERWISE "EULERIZE" your graph:
Strategically "ADD" edges to your graph.
These represent edges you plan to cross more than once.
These edges should eliminate odd vertices so that your "new" graph has a Euler Circuit or Euler Path.

Consider the following model of a street network.



How many odd vertices are there? 8

Does an Euler Circuit exist? NO

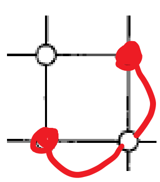
Strategically add edges to eliminate odd vertices.

You will need AT LEAST __4__ edges.

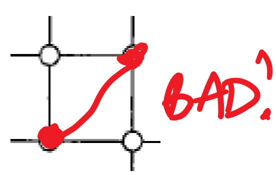
You MAY NOT create new edges! You may only DUPLICATE EXISTING EDGES!

Example:

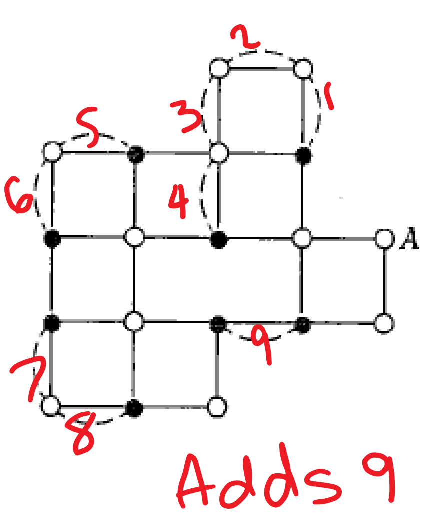
This is OK

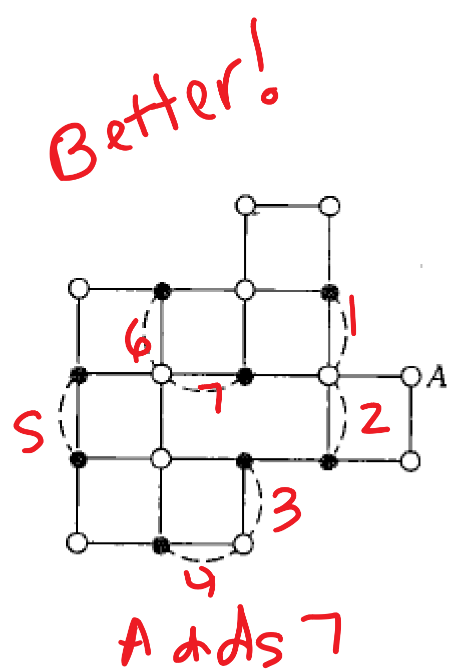


This is NOT OK



Here are a couple of solutions:

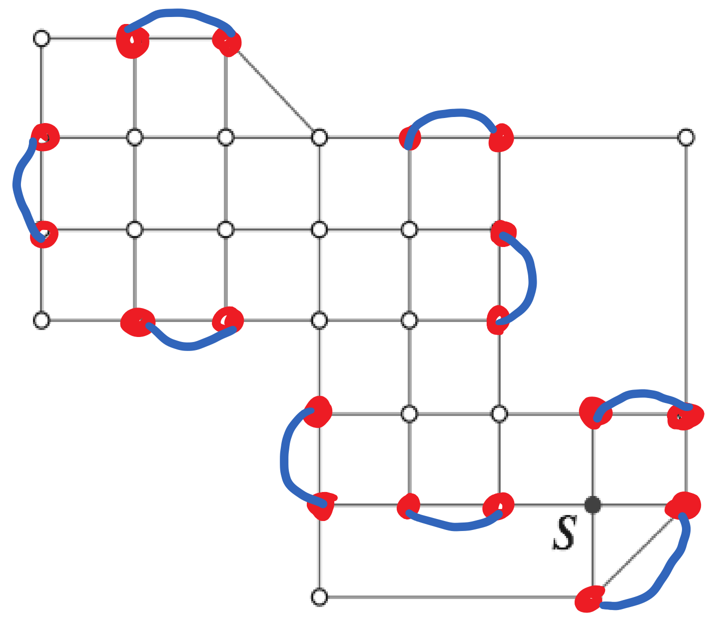




Ex. 1 Find an **optimal eulerization** of the following graph.

18odd

Goal: 9 or more

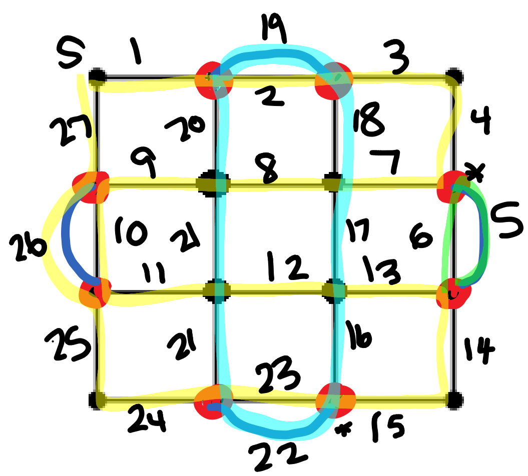


Ex. 2.

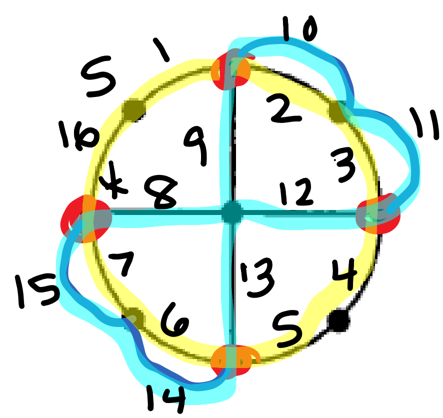
Find an **optimal eulerization** of the following graph.

Then **FIND the Euler Circuit** through the graph by labeling the edges 1, 2, 3… etc. as you would travel them.

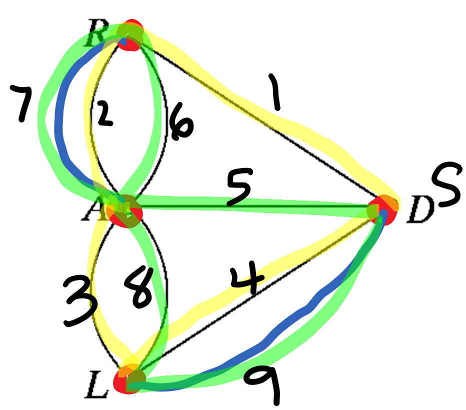
a)



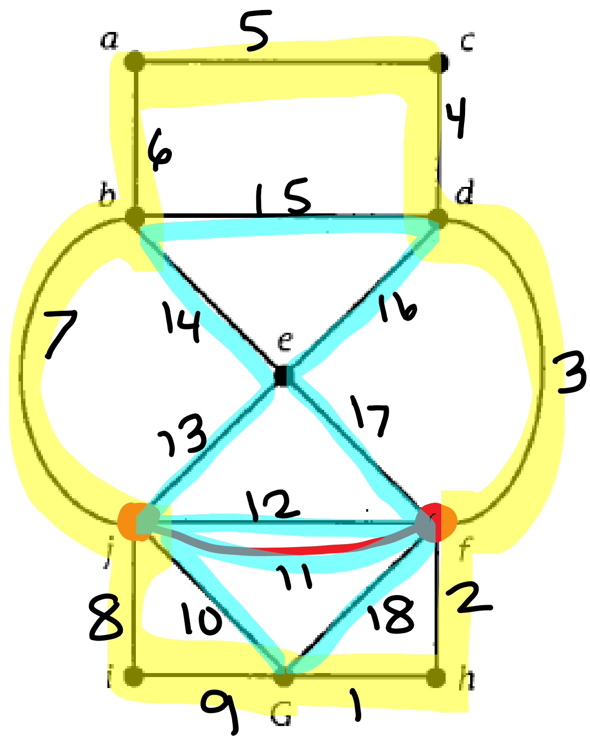
b)



c.



Example 3: The street network of a city can be modeled with a graph in which the vertices represent the street corners and the edges represent the streets. Suppose you are the city street inspector and it is desirable to minimize time and cost by not inspecting the same street more than once.



a. In this graph of the city, is it possible to begin at the garage (G) and inspect each street only once? Will you be back at the garage at the end of the inspection?

No. No Euler Path or Circuit starting at G

b. Find a rout that inspects all street, repeats the least number of edges possible, and returns to the garage.

c. Suppose it takes you approximately 5 minutes to inspect each street. Approximately how long will it take you to inspect the entire street network?

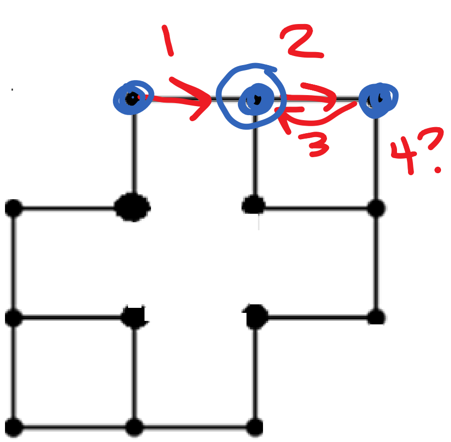
$18 \times 5 = 90$ MIN

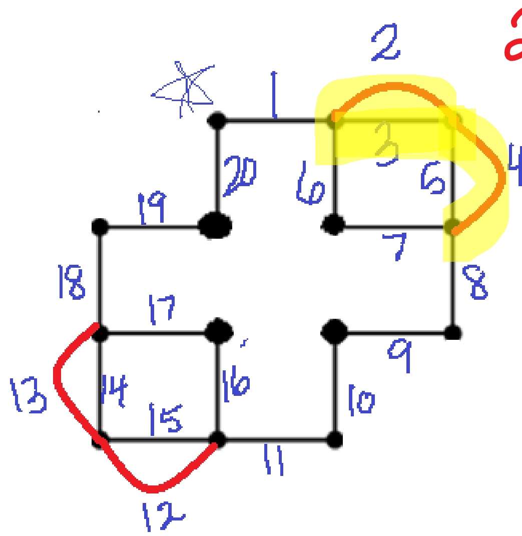
or $1\frac{1}{2}$ hrs

NOTE:
When labeling your edges, it is not a bad idea to have a friend try to follow your work.

For example: What is wrong with this problem?

Find the optimal eulerization of the following graph. Then find the Euler circuit by labeling the edges 1, 2, 3… as you travel them.





2→3→4 is Not possible